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THE CLASS OF MIFRA LIFETIMES AND ITS RELATION TO OTHER CLASSES.(U)

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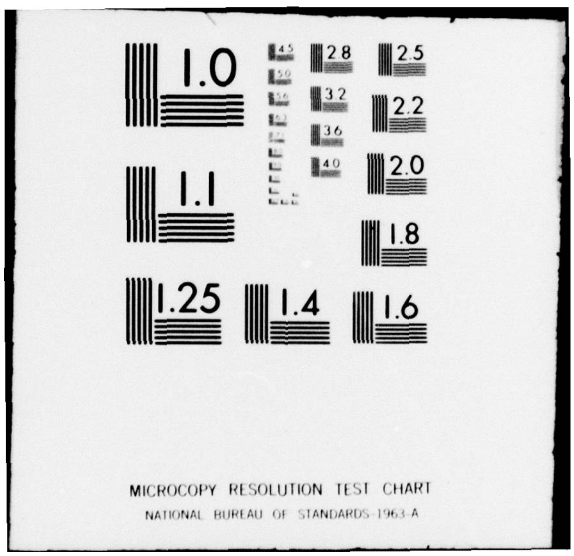
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The Class of MIFRA Lifetimes

and Its Relation to Other Classes

by

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The Class of MIFRA Lifetimes and Its Relation to Other Classes[1]

by

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Abstract

In a previous paper, the authors have introduced a class of multivariate lifetimes (MIFRA) which generalize the univariate lifetimes with increasing failure rate average (IFRA). They have also shown that this class satisfies many fundamental properties. In this paper it is shown that other concepts of multivariate IFRA do not satisfy all of these properties. Relationships between MIFRA and these other concepts are given. Finally positive dependence implications with respect to these classes are also discussed.

AMS 1970 subject classification. Primary: 62N05 Secondary: 60K10

Key words. IFRA, MIFRA, monotone and coherent structure and life functions, positive dependence, association.

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The Class of MIFRA Lifetimes

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1. Introduction. The class of univariate lifetimes with increasing failure rate average (IFRA) has been of great importance in reliability theory. The importance of the class, and properties thereof, are discussed in the text of Barlow and Proschan [1] whose notation and terminology are followed here. A recent development with respect to this class, has been the resolution, by Block and Savits [2], of a long standing problem concerning the closure of this class under convolution.

Several recent papers by Block and Savits [3, 4], and by Esary and Marshall [6] have proposed various multivariate extensions of this univariate class. It is our purpose in the present paper to give the relations among these various concepts and to show that one of these concepts, which was designated MIFRA in Block and Savits [3], is preferable to others. This will be done by showing that among these various extensions only the MIFRA class of distributions satisfies all of the properties which one would reasonably expect for a class of multivariate IFRA distributions. Furthermore dependence properties and the lack thereof for those classes are also discussed.

One deviation which we shall make from the notation of Barlow and Proschan [1] is to call a structure function $\phi(\underline{x})$ monotone if it is increasing in its components and in addition $\phi(\underline{0}) = 0$ and $\phi(\underline{1}) = 1$. Esary and Marshall [5] have called such a function coherent.

We conform to the terminology of Barlow and Proschan [1], and call a structure function coherent (called fully coherent by Esary and Marshall) if it is increasing in its arguments and if all components are essential. The life function τ corresponding to a system ϕ is called monotone (coherent) if ϕ is monotone (coherent). See Esary and Marshall [5] for a discussion of life functions.

2. Multivariate IFRA. Block and Savits [3] have introduced a concept of multivariate IFRA which is given in the following difinition.

Definition 2.1. Let $\underline{T} = (T_1,\ldots,T_m)$ be a nonnegative random vector. The vector $\underline{T}$ is said to be MIFRA iff

$$E^{\alpha}[h(\underline{T})] \leq E[h^{\alpha}(\underline{T}/\alpha)]$$

for all continuous nonnegative increasing functions h and all $0 < \alpha \leq 1$.

Several other possible conditions for multivariate IFRA have been proposed.

Definition 2.2. Let $\underline{T} = (T_1,\ldots,T_m)$ be a nonnegative random vector with survival function $\overline{F}(\underline{t}) = P(\underline{T} > \underline{t})$. The vector $\underline{T}$ is said to satisfy condition _____ if the condition following ______ is satisfied.

$\underline{A}$: $\overline{F}^{\alpha}(\underline{t}) \leq \overline{F}(\alpha\, \underline{t})$ for all $0 < \alpha \leq 1$ and all $\underline{0} \leq \underline{t}$.

$\underline{B}$: $\underline{T}$ is such that each monotone system formed from $\underline{T}$ is univariate IFRA.

$\underline{C}$: $\underline{T}$ is such that there exist independent IFRA random variables $X_1,\ldots,X_k$ and monotone life functions τ_i, $i = 1,\ldots,m$ such that $T_i = \tau_i\,(X_1,\ldots,X_k)$ for $i = 1,\ldots,m$.

$\underline{\Sigma}$: $\underline{T}$ is such that there exist independent IFRA random variables $X_1,\ldots,X_k$ and nonempty sets S_i of $\{1,\ldots,k\}$ such that $T_i = \sum_{j\in S_i} X_j$

for $i = 1,\ldots,m$.

D: $\underline{T}$ is such that there exist independent IFRA random variables $X_1,\ldots,X_k$ and nonempty subsets S_i of $\{1,\ldots,k\}$ such that $T_i = \min_{j\in S_i} X_j$ for $i = 1,\ldots,m$.

E: $\underline{T}$ is such that the minimum of any subfamily of $T_1,\ldots,T_m$ is IFRA.

F: $\underline{T}$ is such that $\min_i a_iT_i$ is IFRA for all $a_i \geq 0$, $i = 1,\ldots,m$.

Conditions A, B, C, D, E, F have been given by Esary and Marshall [6] and condition Σ was given by Block and Savits [4].

3. Relationships Among the Conditions. The following relationships hold between MIFRA and the seven conditions given in Section 2.

$$\begin{tikzcd}
D \arrow[d, Rightarrow] & & \Sigma \arrow[ll, Rightarrow, "?"'] \arrow[ddl, Rightarrow] \\
C \arrow[dr, Rightarrow] & & \\
& \mathrm{MIFRA} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
A(=)F \arrow[dr, Rightarrow] & & B \arrow[dl, Rightarrow] \\
& E &
\end{tikzcd}$$

Figure 1

With the exception of the implication $\Sigma \Rightarrow C$, the above figure is complete, i.e. no more implications are possible. It is not known whether $\Sigma \Rightarrow C$ holds, but we conjecture that it does not. Proofs of the remaining implications and counterexamples will now be given.

Because of results in Esary and Marshall [6] we need only show how MIFRA and Σ compare with concepts A, B, C, D, E and with each other.

3.1 Comparison of MIFRA.

a. $C \Longrightarrow$ MIFRA. This follows from (iii) of Theorem 4.1 of Block and Savits [3].

b. $\Sigma \Longrightarrow$ MIFRA. See (iv) of Theorem 4.1, ibid.

c. MIFRA $\Longrightarrow F$. Apply (P1) and (P5) of Theorem 2.3, ibid.

d. MIFRA $\Longrightarrow B$. This is (P1) of Theorem 2.3, ibid.

e. MIFRA $\not\Longrightarrow \Sigma$. Given in Example 3.3 of Block and Savits [4].

f. MIFRA $\not\Longrightarrow C$. (and MIFRA $\not\Longrightarrow D$). Example 3.2, ibid.

g. $A \not\Longrightarrow$ MIFRA. Since $A \not\Longrightarrow B$.

h. $B \not\Longrightarrow$ MIFRA. Since $B \not\Longrightarrow A$.

3.2 Comparison of Σ.

a. $D \not\Longrightarrow \Sigma$. Example 3.3 of Block and Savits [4].

b. $C \not\Longrightarrow \Sigma$. Since $D \not\Longrightarrow \Sigma$.

c. $\Sigma \not\Longrightarrow D$. Let X_1, X_2, X_3 be absolutely continuous IFRA random variables. Form $Y_1 = X_1 + X_3$ and $Y_2 = X_2 + X_3$. By definition (Y_1, Y_2) satisfies Σ, but by Section 10 of Esary and Marshall [6] (Y_1, Y_2) does not satisfy D.

All other counterexamples and implications, with the exception of $\Sigma \Longrightarrow C$, follow from the above.

4. Properties Relevant to Multivariate IFRA Distributions.

The class $\mathcal{C}$ of MIFRA distributions has been shown by Block and Savits [3] to satisfy the following properties:

(P1): Closure under the formation of monotone systems, i.e. if $(T_1,\ldots,T_n) \in \mathcal{C}$ and $\tau_1,\ldots,\tau_m$ are montone life functions, then $(\tau_1(T_1,\ldots,T_n),\ldots,\tau_m(T_1,\ldots,T_n)) \in \mathcal{C}$.

(P2): Closure under limits in distribution.

(P3): Marginals are in the same class.

(P4): Closure under conjunction of independent sets of lifetimes, i.e. if $(T_1,\ldots,T_n)$ and $(S_1,\ldots,S_m) \in \mathcal{C}$ and are independent, then $(T_1,\ldots,T_n, S_1,\ldots,S_m) \in \mathcal{C}$.

(P5): Closure under scaling, i.e. if $(T_1,\ldots,T_n) \in \mathcal{C}$ and a_i, $i = 1,\ldots,n$, are nonnegative constants, then $(a_1 T_1,\ldots,a_n T_n) \in \mathcal{C}$.

(P6): $\mathcal{C}$ is closed under well defined convolution, i.e. if $(T_1,\ldots,T_n) \in \mathcal{C}$ and $(S_1,\ldots,S_n) \in \mathcal{C}$ and independent, then $(T_1 + S_1,\ldots,T_n + S_n) \in \mathcal{C}$.

It is reasonable that any class of multivariate IFRA distributions should satisfy these conditions. Block and Savits [3] have shown that the MIFRA distributions satisfy these conditions. We will now show that each of the conditions A, B, C, Σ, D, E, F fails to satisfy at least one of these properties.

4.1 A does not satisfy P1. This follows since $A \not\Rightarrow B$.

4.2 B does not satisfy P5. This follows since $B \not\Rightarrow A$.

4.3 C (and D) do not satisfy P5. Let $(T_1, T_2) = (\min(X, Z), \min(Y, Z))$ where X, Y and Z are independent exponential random variables with mean one. For $a_1 \neq a_2$, assume $(a_1 T_1, a_2 T_2) = (\tau_1(X_1,\ldots,X_k), \tau_2(X_1,\ldots,X_k))$ where τ_1, τ_2 are monotone life functions and $X_1,\ldots,X_k$ are independent IFRA lifetimes. It follows from Remark 2.2a of Block and Savits [4] that there exist independent exponential random variables U, V, W such that $(a_1 T_1, a_2 T_2) = (\min(U, W), \min(V, W))$. But the conditions $0 = P(a T_1 = a_2 T_2)$ and

$P(\min(U, W) = \min(V, W)) > 0$ are not compatible.

4.4 <u>Σ does not satisfy P1</u>. Let X and Y be independent exponential lifetimes. Define $\tau_1(X, Y) = \min(X, Y)$ and $\tau_2(X, Y) = Y$ and assume that $(\tau_1, \tau_2) = (U + W, V + W)$ where U, V, and W are independent IFRA lifetimes. Now by Theorem 2.8 of Block and Savits [4], one of V and W is exponential and one is concentrated at 0. If W is exponential, since $P(\min(X, Y) \leq Y) = 1$ it follows that $P(U = 0) = 1$ and so $P(\min(X, Y) = Y) = 1$ which is impossible if X and Y are independent exponentials. If V is exponential, then $P(W = 0) = 1$ so that $\min(X, Y)$ and Y are independent, again an impossibility.

4.5 <u>D does not satisfy P6</u>. Let X, Y, and Z be independent absolutely continuous IFRA lifetimes. Then both (X, Y) and (Z, Z) are trivially in D. However, if $(X, Y) + (Z, Z) = (X + Z, Y + Z)$ was in D, then by Section 10 of Esary and Marshall [6] $X + Z$ and $Y + Z$ would be independent, but they can't be.

4.6 <u>E does not satisfy P1</u>. This follows since $E \not\Rightarrow B$.

4.7 <u>F does not satisfy P1</u>. This follows since $A \not\Leftrightarrow F$.

5. <u>Positive Dependence</u>. The first published definition of a class of multivariate nonparametric reliability distributions was Harris' [7] definition for multivariate increasing hazard rate. This definition included a type of positive dependence (i.e. right corner set increasing). See Barlow and Proschan [1] for a discussion of various types of positive dependence. Subsequent definitions have not included such

assumptions. The opinion which is now generally held is that the various concepts of positive dependence are not intimately related to useful definitions for nonparametric multivariate life classes. In other words, if a multivariate lifetime has an increasing failure rate or failure rate average, then it need not follow that the lifetime be positively dependent in some sense. In fact, if such a definition implies positive dependence, then it is probably too strong. Examples of such definitions are conditions C, D and Σ which are easily shown to imply association. We will show that the more useful definitions $A(\equiv F)$, B, E and especially MIFRA do not imply even positive quadrant dependence, which is one of the weaker types of positive dependence.

5.1 A and E $\not\Rightarrow$ Positive Quadrant Dependence.

Clearly $\overline{F}(t_1, t_2) = P(T_1 > t_1, T_2 > t_2) = \exp(-t_1-t_2-t_1t_2)$ satisfies A and E, but $\overline{F}(t_1, t_2) \leq P(T_1 > t_1) P(T_2 > t_2)$.

5.2 MIFRA and B $\not\Rightarrow$ Positive Quadrant Dependence.

Consider $(T_1, T_2) = (U, 1-U)$ where U is a uniform distribution on the unit interval. Clearly $\overline{F}(t_1, t_2) \leq P(T_1 > t_1) P(T_2 > t_2)$, but T_1, T_2, min (t_1, T_2) and max (T_1, T_2) have a univariate uniform distribution and so are IFRA. Thus B is satisfied. Furthermore, Theorem 3.5 of Block and Savits [3] gives that (T_1, T_2) is MIFRA if the indicator function of every fundamental upper domain in $\mathbb{R}_2^+$ satisfies the inequality of Definition 2.1. A set A is a fundamental upper domain if it has the form in Diagram 5.1 below where $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$.

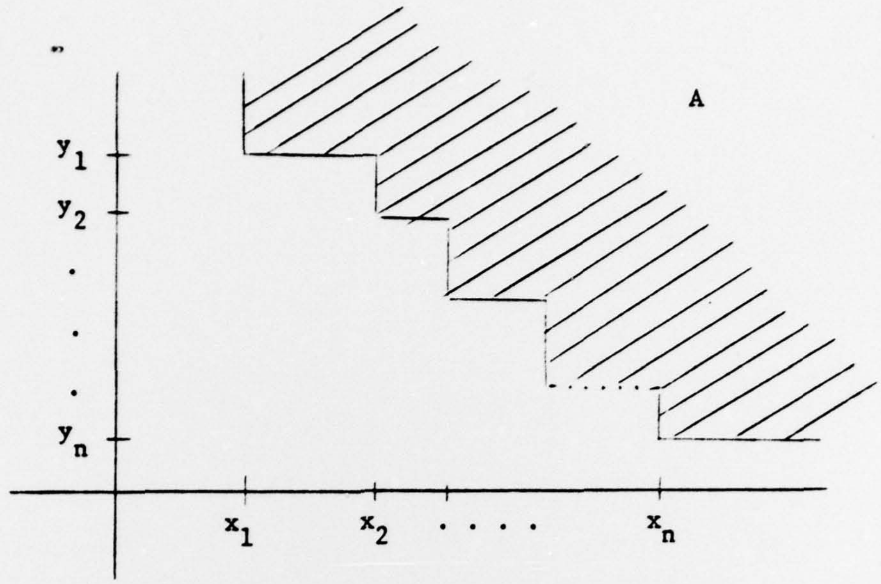

Diagram 5.1

From the diagram,

$$\{(T_1, T_2) \in A\} = \bigcup_{i=1}^{n} \{T_1 > x_i,\ T_2 > y_i\} = \bigcup_{i=1}^{n} \{x_i < U < 1 - y_i\}.$$

and

$$\{(T_1/\alpha,\ T_2/\alpha) \in A\} = \bigcup_{i=1}^{n} \{\alpha x_i < U < 1 - \alpha y_i\}.\ \ \text{Let}$$

$I = \{i:\ x_i + y_i < 1\}$, $J = \{j:\ \alpha x_j + \alpha y_j < 1\}$. Since $0 < \alpha \leq 1$, $I \subseteq J$. Then

$$E[I_A(T_1, T_2)] = P[\textstyle\bigcup_{i \in I}\{x_i < U < 1 - y_i\}] \quad \text{and}$$

$$E^{1/\alpha}[I_A^{\alpha}(T_1/\alpha,\ T_2/\alpha)] = P^{1/\alpha}[\textstyle\bigcup_{j \in J}\{\alpha x_j < U < 1 - \alpha y_j\}]$$

$$\geq P^{1/\alpha}[\textstyle\bigcup_{i \in I}\{\alpha x_i < U < 1 - \alpha y_i\}].$$

By renumbering if necessary, we may assume without loss of generality that $I = \{1, 2, \ldots, p\}$. Now define $K = \{2 \leq k \leq p:\ \alpha x_k \geq 1 - \alpha v_{k-1}\} = \{k_1 < k_2 < \ldots < k_r\}$ and set $k_o = 1$ and $k_{r+1} = p + 1$. Then

$$\bigcup_{i \in I}\{\alpha x_i < U < 1 - \alpha y_i\} = \bigcup_{l=1}^{r+1} \{\alpha x_{k_{l-1}} < U < 1 - \alpha y_{k_l - 1}\}$$

and these latter sets are disjoint intervals. It follows from Minkowski's inequality for $0 < \alpha \leq 1$ that

$$P^{1/\alpha}[\bigcup_{i \in I}\{\alpha x_i < U < 1 - \alpha y_i\}] = \{\sum_{l=1}^{r+1} P\{\alpha x_{k_{l-1}} < U < 1 - \alpha y_{k_l - 1}\}\}^{1/\alpha}$$

$$\geq \sum_{l=1}^{r+1} P^{1/\alpha}\{\alpha x_{k_{l-1}} < U < 1 - \alpha y_{k_l - 1}\}.$$

Since $\bigcup_{i \in I}\{x_i < U < 1 - y_i\} = \bigcup_{l=1}^{r+1} \bigcup_{k_{l-1} \leq i \leq k_l - 1} \{x_i < U < 1 - y_i\}$, it suffices to show that for $l = 1,\ldots,r + 1$

$$P[\bigcup_{k_{l-1} \leq i \leq k_l - 1}\{x_i < U < 1 - y_i\}] \leq P^{1/\alpha}\{\alpha x_{k_{l-1}} < U < 1 - \alpha y_{k_l - 1}\} .$$

But the union on the left hand side in contained in the interval $\{x_{k_{l-1}} < U < 1 - y_{k_l - 1}\}$ and $x_{k_{l-1}} + y_{k_l - 1} < 1$ since for $k_{l-1} \leq i \leq k_l - 1$, $i \in I$. And so we are done if we show that $1 - t \leq (1 - \alpha t)^{1/\alpha}$ for $0 \leq t \leq 1$, $0 < \alpha \leq 1$. This last inequaltiy is, however, easily verified.

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